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REPETITION IN TEACHING MATHEMATICS

***Abstract**. The paper reveals the essence and main characteristics of the concept of repetition, in particular in teaching mathematics, its role and importance for optimizing the learning process. Various possibilities of performing repetition in teaching mathematics in the secondary school are presented. The emphasis is placed on revealing important links between the knowledge used in different ways to solve the suggested problems, the discovery of which is based on various ideas and methods for finding out solutions.*
***Keywords**: repetition, method, idea, way, solving problems.*

One of the main goals of the National Development Program "Bulgaria 2020" is to improve the quality of education. The quality of education in secondary school is related to the acquisition of lasting knowledge, skills and habits by students and developing key competencies in them. The principle of durability of knowledge, skills and habits requires that their acquisition is characterized by meaningfulness and depth, that students be able to reproduce and apply the essentials of the studied material in different situations. When the student learns the material consciously, systematically, accessible and with interest, the acquired knowledge is characterized by durability and depth. Permanent mastery of knowledge, skills and habits is of great importance for the development of all aspects of personality. In-depth study of the material requires a repeated return to it, and its consideration taking into account the different connections and contexts.

Repetition is essential for the lasting memorization of the studied material. John Comenius says, that “it is impossible to achieve perfect training without conducting more frequent and especially masterfully designed repetitions and exercises” (Comenius, 2007)

M. Andreev believes that repetition is an important means for learning and confirming students' knowledge. According to him "the primary, current and generalizing repetition are of the greatest importance. The primary repetition is carried out in the same way as the new learning material is perceived. The current repetition consists of constant reproduction or turning to the knowledge, that is previously learned, in the same plan in which it was received, but in different forms. To be this repetition effective, it's important to include the repeated knowledge in solving new creative problems. The generalizing repetition also uses the accepted new knowledge, but in a different logical plan – they obey a more general idea and lead to a higher synthesis of knowledge." (Andreev, 1996)

A. Mavrodiev believes that there is a very large difference "between the spontaneous arbitrary repetition and the repetition based on rational principles. The irrational approach, even with frequently repetition, leads only to superficial confirming. The use of rational methods makes it possible to achieve both more efficient confirmation of knowledge and high memory productivity, even with a relatively small number of repetitions." (Mavrodiev, 2011). In our opinion, the use of mnemonic rules can refer to the rational methods for effective memorization and confirmation of information.

Memory will be more effective if a constant and accurate distinction is made (with the help of the teacher) between what: first – must be remembered and always known; second - what only needs to be read in order the student to have a general view of it, which in time will only need to be refreshed; and third, what students only need to get acquainted with and know where they can refer to the certain issue later.

Both memorizing and repeating the information remembered is a cognitive process. They lead to a common goal - more complete and lasting acquisition of knowledge. That's why, it is necessary to carry them out selectively and purposefully. Due to the purposefulness of repetition, its essence is expressed in active thinking, because in the process of repeating the relevant learning content, the learner can find new relations and new properties in it, can associate new material with older knowledge or with knowledge acquired in the period between primary memorization and repetition. In other words, in the process of purposeful conscious repetition, the learner *comprehends the new information actively and rethinks the old one*. Therefore, through conscious repetition a higher effect of memorization is achieved. It is recommended that the repetition is performed at regular intervals which are not too long. Each trainee, if possible, should decide for himself what these intervals should be. It is also necessary to separate this knowledge, which must be remembered for a short period of time, from that which needs to remain in the memory for a longer time. In addition, the knowledge that is need to be remembered for a lifetime must be highlighted.

"The peculiarity of repetition in comparison with perception, concentration and memorization consists in the *repeated* course of the process of memorization from perception (through concentration) to new memorization. Repetition increases the intensity of these phases of memory." (Mavrodiev, 2011). In this way, the updated information is included more and more fully and comprehensively in the general information network of the subject and as a result it is fixed more permanently in his memory.

Since the definitions of the scientific concepts, included in the mathematics curricula of different classes, are usually precisely presented in the textbooks, their *primary repetition* is aimed at remembering them in the form in which they had been formulated during their perception and initial mastery. This is important, because it contributes to their purposeful memorization, with a view to further application in defining new concepts, proving theorems and solving problems. That's why, except of the primary repetitions, the so-called *current repetitions*, which are performed in the process of formation of new concepts and their application, are more and more finding place in the learning process.

In addition to current repetitions, special repetitions, called *thematic repetitions*, are organized and performed when a certain section (or a topic) is completed. They are carried out in the *repetition* and *summarizing* lessons. The role and the purpose of these repetitions, performed in these two types of lessons, is to systematize the knowledge of students from the studied section (topic), to reach to deeper connections between the concepts considered in it, and also from previous topics in the same class, and even from previous years, with a view to create a comprehensive system of mathematical knowledge. That is why repetition and summary lessons are essential for teaching mathematics. Therefore, a number of authors have dedicated their works especifically to this topic, such as the book "Repetition in the Teaching Mathematics" (Portev, 1992). The author states that the repetition can be at the beginning of the school year (initial repetition), thematic and final. In these types of repetitions, various forms and methods can be used, such as: frontal talk, summary talk, independent work of students, exercises, lessons-seminars, practical work, etc. For this purpose, it is appropriate to use visual aids – a screen or a paper, as well as models.

Our practical experience shows that the system of exercises designed to confirm a learning content by repeating it in different versions of implementation, are the more effective, the more diverse efforts they require from students in their achievement. We will present some examples from different sections of the school course in mathematics, which we introduce to our university students preparing for teachers in mathematics.

Example 1. We show students (on screen) graphs of functions, they have previously studied and ask them the following problems and questions:

1. Recall the names of the functions which graphs are displayed on the screen.
2. Are these functions have their opposite ones?
3. Indicate the intervals in which different functions increase or decrease.

Example 2. After studying the topics of logarithmic function, logarithmic equation and logarithmic inequalities, it is advisable the following exercises to be repeated. The graph of a specific logarithmic function is displayed on a screen and the following tasks are set:

1. Write down analytically the function, which graph is presented on the screen.
2. Write down what equations or inequalities that can be solved, using the graph of this function.
3. Write down the solutions of some of the inequalities that you have drawn according to the given graph.

Using IT we can repeat the transformation of the graph of a function, graphical solution of equation "..." or inequality "...", etc.

Example 3. After studying the topic "Midsegment in a triangle", students can be given the following figure (fig. 1.) and they can be set the following tasks:

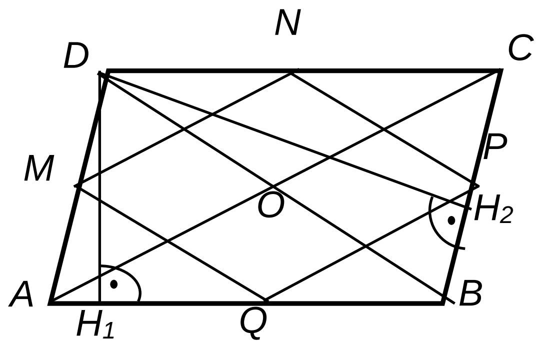

Fig. 1.

1. Prove that the perimeter of $MNPQ$ is equal to the sum of the lengths of the diagonals of the parallelogram $ABCD$.

2. Prove that $\measuredangle CDH_2 = \measuredangle ADH_1$ (DH_1 and DH_2 are heights of the parallelogram $ABCD$).

3. Prove that the areas of the triangles AOD, DOC, COB and AOB are equal.

Example 4. Repetition can be done by drawing up diagrams by students – semi-finished diagrams, which have to be filled in the lesson. For example, the following diagrams can be used when studying different types of numbers:

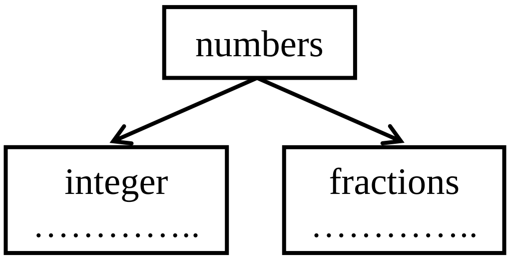

Diagram 1.

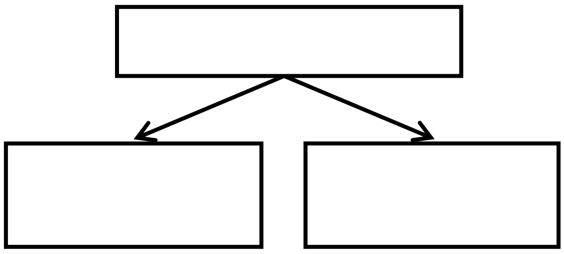

Diagram 2.

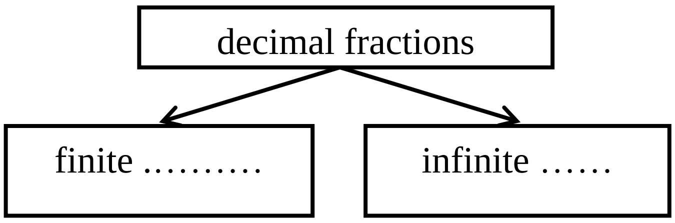

Diagram 3.

In each of the diagrams, the students must write numbers (on the places marked with dots), corresponding to the type of number, written in the box. Diagrams can also be used in generalizing repetitions in order to find interrelationships between different types of numbers, which contributes to a more conscious understanding of these types of numbers, as well as the relationships established between them. Through such and similar diagrams, students can also find different connections between other studied concepts.

The use of such diagrams helps to develop students' thinking and at the same time arouses their interest in the studied material. For example, when studying the theme "Quadrilateral", it would be useful in the generalized repetition of the terms quadrilateral, parallelogram, trapezium, rectangle, rhombus and square, to illustrate their relationships (presented in diagram 4). The diagram that students will make requires of them more mental activity, as they have to separate one set of objects from another on a certain feature.

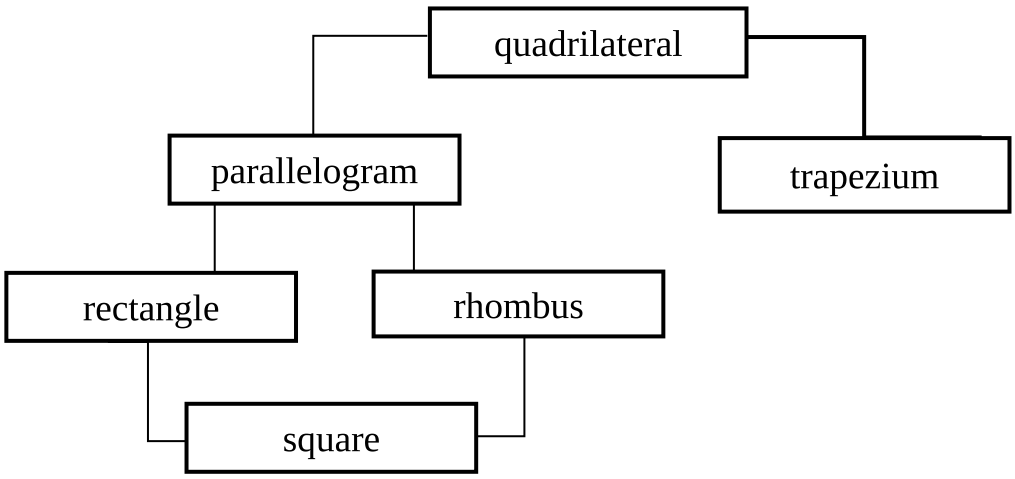

Diagram 4.

Such diagrams can be used in other themes from the school course in mathematics. For example, in the theme “Rational expressions” diagram 5 can be used.

Using such diagrams, the so-called generalizing repetition is carried out, in which not only the most essential facts, concepts, skills are reproduced, but also logical connections between the concepts are established. The material is reconsidered from different points of view, as a result of which the quality of the mastered material increases, the knowledge is systematized and all this helps to develop the mental activity of students.

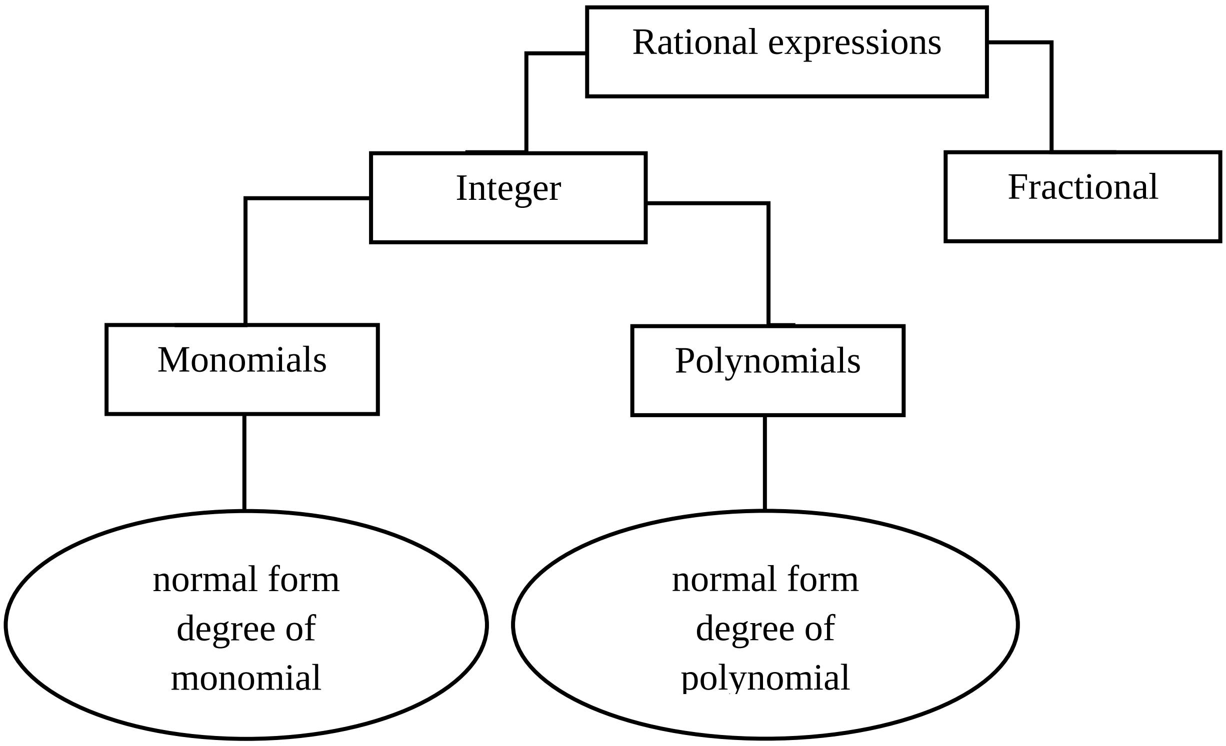

Diagram 5.

Example 5. After studying the figure of a parallelogram and types of parallelograms, in order to confirm and master their properties more deeply, it is appropriate to use the following task when repeating the topic.

Match the name of the quadrilateral with its properties:

QUADRILATERAL
Rhombus
Parallelogram
Square
Rectangle

PROPERTIES
a) The diagonals have a common midpoint
b) The diagonals are equal
c) The opposite sides are equal
d) The diagonals are mutually perpendicular

Repetition of knowledge can also be done by solving problems, especially such in which different ways can be applied, using a variety of ideas and knowledge from different sections of the school course in mathematics. To illustrate it, we will suggest some specific problems. For example, while repeating the knowledge of intersecting lines and a tangent line to a circle from a point outside the circle and the properties of the bisector of an interior angle in a triangle, the following problem can be solved.

Problem 1. From point M to a circle k a tangent line MA and an intersecting line, which crosse the circle in points B and C are constructed. The bisector of $\measuredangle AMC$ crosses the chords AB and AC respectively in point P and point E (fig. 2).

a) Prove that $AP = AE$; b) Find out AP, if $PB = 4$ cm, $EC = 9$ cm.

Solution:

a) In order to prove that $AP = AE$, we turn to the idea of finding that ΔAPE is isosceles. For this purpose, it's enough to prove that $\measuredangle P$ and $\measuredangle E$ in it are equal. For a brief description let's denote $\measuredangle APE = \measuredangle 1$, $\measuredangle AEP = \measuredangle 2$, $\measuredangle AME = \measuredangle EMC = \alpha$, $\measuredangle MAB = \beta$ и $\measuredangle ACM = \gamma$ (fig. 2).

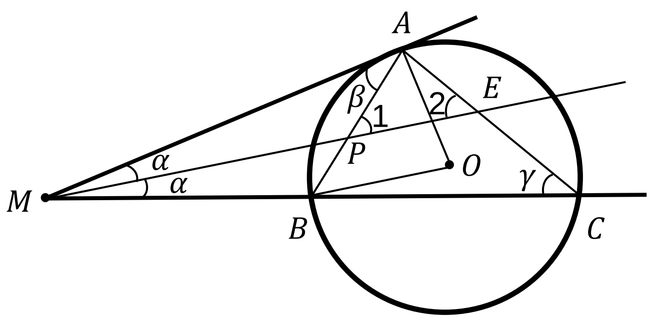

Fig. 2.

Since $\measuredangle APE$ is external to ΔAMP, then $\measuredangle 1 = \alpha + \beta$. Similarly $\measuredangle AEP$ is external to ΔMEC, then $\measuredangle 2 = \alpha + \gamma$.

Because $\gamma = \frac{1}{2}\measuredangle AOB = \frac{1}{2}(180^0 - 2.\measuredangle BAO) = 90^0 - \measuredangle BAO = \beta$, then $\gamma = \beta$. That's why $\measuredangle 1 = \measuredangle 2$. Therefore ΔAPE is isosceles, i. e. $AP = AE$.

b) Since ME is a bisector both for ΔAMB and ΔAMC, then the following proportions are fulfilled: $\frac{AE}{AM} = \frac{EC}{CM}$ and $\frac{AP}{AM} = \frac{PB}{MB}$. Let's multiply their left and their right sides and receive $\frac{AE.AP}{AM^2} = \frac{EC.PB}{CM.MB}$.

Because AM is a tangent to the circle, then $AM^2 = CM.MB$, which means that the dominators of the equation above are equal. Then the numerators are also equal, i.e. $AE.AP = EC.PB$. But in a) we have proved that $AP = AE$. That's why the last equality takes the following form: $AP^2 = EC.PB$. Therefore $AP = \sqrt{9.4} = 6\ cm$.

In a repetition lesson it is appropriate to suggest a problem which is not very complicated, but allows to find different ways to solve it using a variety of knowledge. Such, for example, is the following problem.

Problem 2. Solve the system

$$\left| \begin{array}{l} x + y = \frac{\pi}{3} \\ tgx.tgy = \frac{1}{3} \end{array} \right., x \neq (2k+1)\frac{\pi}{2}, y \neq (2k+1)\frac{\pi}{2}, \text{ a } k \in \mathbb{Z}.$$

This system can be solved in different ways.

I way: By substitution.

Let's express $y = \frac{\pi}{3} - x$ from the first equation. Then the second equation takes the form $tgx.tg\left(\frac{\pi}{3} - x\right) = \frac{1}{3}$. Applying the tangent *subtraction* formula the following rational equation is obtained: $tgx.\frac{tg\frac{\pi}{3} - tgx}{1 + tg\frac{\pi}{3}.tgx} = \frac{1}{3}$, i.e. $tgx.\frac{\sqrt{3} - tgx}{1 + \sqrt{3}.tgx} = \frac{1}{3}$, where

$1 + \sqrt{3}.tgx \neq 0$. The last restriction can be written as $tgx \neq -\frac{\sqrt{3}}{3}$.

By the substitution $tgx = t$ and getting rid of the denominator we receive $3t\left(\sqrt{3} - t\right) = 1 + \sqrt{3}t \iff 3t^2 - 2\sqrt{3}t + 1 = 0 \iff (\sqrt{3}t - 1)^2 = 0$.

Therefore $t = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, i.e. $tgx = \frac{\sqrt{3}}{3}$, whence that $x = \frac{\pi}{6} + k\pi$, and $y = \frac{\pi}{6} - k\pi$, where $k \in \mathbb{Z}$.

II way: From $x + y = \frac{\pi}{3}$ follows, that $tg(x + y) = tg\frac{\pi}{3}$, i.e. $tg(x + y) = \sqrt{3}$. On the other side, $tg(x + y) = \frac{tgx+tgy}{1-tgx.tgy}$. Therefore, from the last two equalities we receive the trigonometric equation with two variables $\frac{tgx+tgy}{1-tgx.tgy} = \sqrt{3}$.

Let's substitute $tgx = u$ and $tgy = v$ and create an algebraic system with the last equation and the second equation in the given system, namely:

$$\left| \begin{array}{l} \frac{u+v}{1-u.v} = \sqrt{3} \\ u.v = \frac{1}{3} \end{array} \right. \iff \left| \begin{array}{l} u + v = \frac{2\sqrt{3}}{3} \\ u.v = \frac{1}{3} \end{array} \right. .$$

Solving the systems, it is received that $u = \frac{\sqrt{3}}{3}$ and $v = \frac{\sqrt{3}}{3}$, whence $tgx = \frac{\sqrt{3}}{3}$ and $tgy = \frac{\sqrt{3}}{3}$. From these basic trigonometric equations there are found x and y: $x = \frac{\pi}{6} + k\pi, y = \frac{\pi}{6} - k\pi$, where $k \in \mathbb{Z}$.

As it can be seen from the solutions of problem 2, finding out different ways for solving problems from trigonometry, is reduced, by applying different trigonometric formulas or substitutions, to the use of different algebraic knowledge – solving a quadratic equation or a system of equations. In this way, different knowledge from the school course of mathematics is repeated and confirmed.

Problem 3. Solve the equation $2^{x+1} = 3^x$.

Solution: Since the bases of the exponentiations are different and we have just one expression in left and right sides (which take only positive values), we can logarithmize both sides on one and the same base. It can be 10 or one of the two bases in the equation.

I way: Let's apply the formula for logarithmizing an exponentiation, namely $lga^n = n.lga$. Then the given equation takes the following form

$$lg2^{x+1} = lg3^x \Leftrightarrow (x+1)lg2 = xlg3.$$

We remove brackets and find out x.

$$xlg2 + lg2 = xlg3 \Leftrightarrow xlg3 - xlg2 = lg2 \Leftrightarrow x(lg3 - lg2) = lg2 \Leftrightarrow$$

$x.lg\frac{3}{2} = lg2$ or $x.lg1{,}5 = lg2$, whence $x = \frac{lg2}{lg1{,}5}$.

The values of the logarithms can be taken from the logarithmic table and the value of the last fraction calculated, but this is usually not required.

II way: Let's logarithmize both sides of the equation at base 2.

$$log_2 2^{x+1} = log_2 3^x \Leftrightarrow (x+1).log_2 2 = x.log_2 3 \Leftrightarrow$$

$$\Leftrightarrow x + 1 = x.log_2 3 \Leftrightarrow x.(log_2 3 - 1) = 1 \Leftrightarrow$$

$$x = \frac{1}{log_2 3 - 1} = \frac{1}{log_2 3 - log_2 2} = \frac{1}{log_2 1{,}5} = \frac{1}{\frac{\lg 1{,}5}{\lg 2}} = \frac{lg2}{lg1{,}5}.$$

Note: In this way of solving, the result, although it can be left in the form $x = \frac{1}{log_2 1{,}5}$, since there are no logarithm tables with base 2, it is usually presented by logarithms with a decimal base, i.e. $x = \frac{lg2}{lg1{,}5}$.

III way: We can do the same if logarithmized at base 3, i.e.

$$log_3 2^{x+1} = log_3 3^x \Leftrightarrow (x+1).log_3 2 = x$$

$$\Leftrightarrow x.(log_3 2 - 1) = -log_3 2 \;\Leftrightarrow$$

$$x = \frac{-log_3 2}{log_3 2-1} = \frac{log_3 2}{1-log_3 2} = \frac{log_3 2}{log_3 3-log_3 2} = \frac{log_3 2}{log_3 1{,}5} = \frac{\frac{lg2}{lg3}}{\frac{lg1{,}5}{lg3}} = \frac{lg2}{lg1{,}5}\,.$$

The note given in II way also applies here. As it can be seen, these solutions make also possible to repeat and use different knowledge of logarithmic operations creatively.

Problem 4. The base of an isosceles triangle is 12 cm and its leg is 10 cm. Find out the radius of the inscribed circle (fig. 3).

Solution:

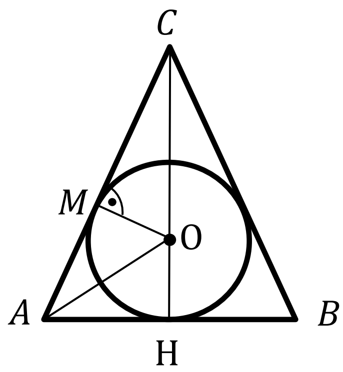

Fig. 3.

I way: In order to find the radius of the inscribed circle, it is appropriate to include it as an element in a suitable triangle. Let’s point O be the center of the inscribed circle. Let’s look at the tringles ΔAHC and ΔOMC. They are similar (by AA (I) criterion). Therefore $\frac{AC}{OC} = \frac{CH}{MC} = \frac{AH}{OM}$.

Because ΔABC is an isosceles, its bisector CH is both an altitude and a median. Therefore, $CH = \sqrt{AC^2 - AH^2}$, i.e. $CH = 8\,cm$, and $CO = 8 - r$. By substituting in the equality $\frac{AC}{OC} = \frac{AH}{OM}$, the following equation is obtained $\frac{10}{8-r} = \frac{6}{r}$, from where it is calculated that $r = 3\,cm$.

II way: The formula about the area of a triangle $S = p.r$ can also be used, where p is the semiperimeter of ΔABC.

On the other hand, $S = \frac{1}{2}.AB.CH = 48\ cm^2$. Then the equation $16.r = 48$ is obtained, whence $r = 3\ cm$.

III way: Let AO is a bisector in ΔAHC. From the property of the bisector, it follows that $\frac{OH}{OC} = \frac{AH}{AC} \Longrightarrow \frac{r}{8-r} = \frac{6}{10} \Longrightarrow r = 3\ cm$.

Solving this problem, various ideas have also been applied, which help to repeat and confirm knowledge from different areas of the planimetry of the triangle.

Problem 5. On the extension of side AC of the equilateral triangle ΔABC a point M is marked, so that $AC = CM$ (fig. 4). Find out: a) the angles of ΔBCM; b) the segment BM, if $AB = 4\sqrt{3}\ cm$.

Solution: For the solution of a) we suggest the following ways.

I way:

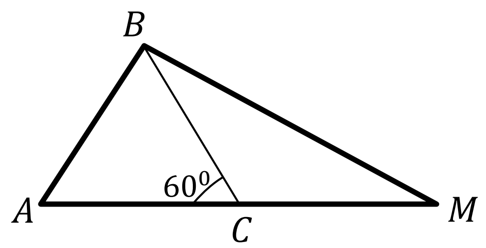

Fig. 4.

Since ΔABC is equilateral, then its angles are equal to 60^0. But $\measuredangle ACB$ and $\measuredangle MCB$ are adjacent corners (fig. 4). Therefore, from the property of adjacent angles it follows, that $\measuredangle BCM = 180^0 - 60^0 = 120^0$. Since $\measuredangle ABM$ is the largest angle in ΔABM, then AM is its largest side. But from $BC = AC = CM$ follows, that the median BC in ΔABM is $BC = \frac{1}{2}AM$. Therefore ΔABM is a right-angle one, as $\measuredangle ABM = 90^0$. Then, from the theorem about the sum of the angles in the triangle is obtained, that $\measuredangle M = 180^0 - 60^0 - 90^0 = 30^0$. To find $\measuredangle CBM$ we can either used ΔBCM, the other two angles of which are known, or the fact that $\measuredangle CBM$ is a part of the right angle ABM, the rest part of which is known. So, we have that $\measuredangle CBM = \measuredangle ABM - \measuredangle ABC = 90^0 - 60^0 = 30^0$.

II way:

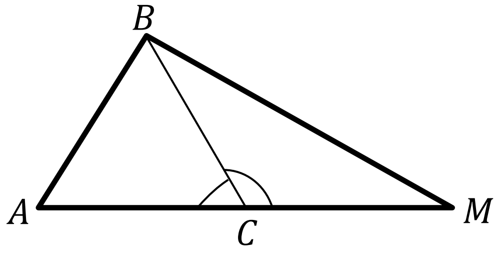

Fig. 5.

Because $\measuredangle BCM$ is an outside corner for ΔABC (fig. 5.), it is fulfilled that $\measuredangle BCM = \measuredangle BAC + \measuredangle ABC,$ i.e. $\measuredangle BCM = 120^0$. Since $AC = CM$ (by given) and $AC = BC$ (ΔABC – equilateral), then $BC = CM,$ which means that ΔBCM is an isosceles one. Therefore, $\measuredangle CBM = \measuredangle CMB = \frac{1}{2}(180^0 - 120^0) = 30^0$.

III way: Let's denote the size of $\measuredangle M$ in the isosceles triangle ΔBCM with x, then $\measuredangle ABM = 60^0 + x$. By the theorem about the sum of angles in ΔABM, we receive, that $60^0 + 60^0 + x + x = 180^0$. From the last equation we find, that $x = 30^0$. Therefore, in ΔBCM we have $\measuredangle BCM = 180^0 - 30^0 - 30^0 = 120^0$.

In condition b) different methods can also be applied.

I way: Because $BC = AB = AC = CM = 4\sqrt{3}$ and $\measuredangle BCM = 120^0$ (an outside corner for ΔABC), then by the cosine theorem for ΔBCM (fig. 6) we have $BM^2 = BC^2 + CM^2 - 2.BC.CM.cos\measuredangle BCM$. After replacing the known elements and calculating, the following result is obtained $BM = 12\ cm$.

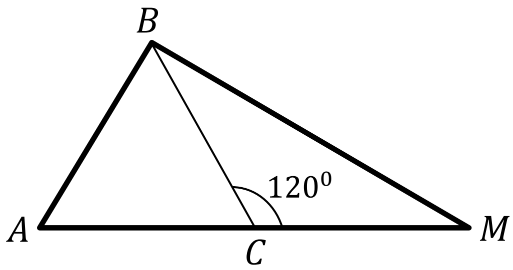

Fig. 6.

II way: Let construct the altitude BH of ΔABC (fig. 7). Then, obviously $HM = HC + CM = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$. By Pythagoras' theorem for ΔAHB we find, that $BH = 6$ см. In the same way from ΔBHM we receive, that $BM = 12\ cm$.

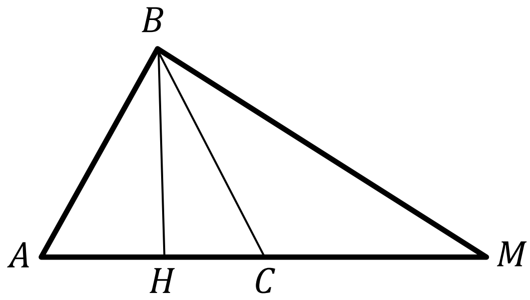

Fig. 7.

III way: In ΔABM point C is equidistant from the vertexes A, B and M. Therefore, point C is a center of the circumscribed circle and its radius R is equal to the segment $AC = 4\sqrt{3}\ cm$. Then, applying the sine theorem for ΔABM, in which $\measuredangle A = 60^0$, we receive, that $\frac{BM}{sin\,60^0} = 2R$. From here we find, that $BM = 2.4\sqrt{3}.sin\,60^0$, i.e. $BM = 12\ cm$.

The presented ways for solving this problem do not exhaust all possibilities. In this way, conditions, so each student can suggest different ideas for solving the problem, using different knowledge, are created: the theorem about the sum of the angles in a triangle; the property of angles in an isosceles triangle; Pythagoras' theorem; the cosine theorem; the sine theorem, etc. Thus, the repetition of various knowledge of geometry, which is used in solving the problem, is carried out, which contributes to its fixation in the memory of the learner and for its deeper assimilation.

Problem 6. A parallelogram $ABCD$ is given, which altitudes from the vertex D are $DH = 2\ cm$ and $DP = 4\ cm$, and $\measuredangle HDP = 30^0$. Find out the sides and the diagonals of the parallelogram.

Solution. When this problem is solved by the students on their lone, they usually draw the parallelogram as the one, shown on the fig. 8., which, however, does not reflect adequately to the given geometric situation. Why does it happen so? Probably they can not consider how the given $\measuredangle HDP = 30^0$ affects the location of the heels of the altitudes from the vertex D.

Of course, this is not obvious to them, or even to the university students, and perhaps to some teachers. To make a reliable drawing, it is necessary to extract additional information from the given in the problem, which contributes not only to develop critical thinking, but also to show creative sense. For this purpose, let us conduct (though on the above drawing) the following considerations.

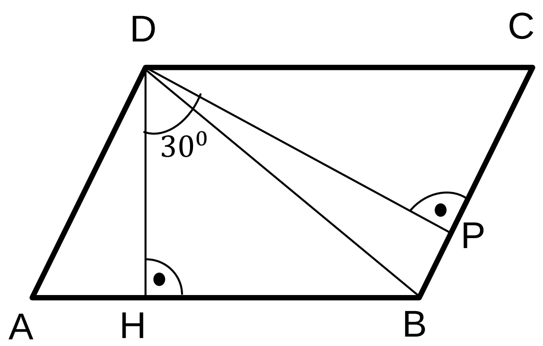

Fig. 8.

Since $\measuredangle HDC = 90^0$, and $\measuredangle HDP = 30^0$, then $\measuredangle PDC = 60^0$. But ΔDPC is a right-angle one, then $\measuredangle DCP = 30^0$, i.e. $\measuredangle DCP < \measuredangle PDC$, and on fig. 8 is the opposite.

Which is wrong – the reasonings or the drawing? The reasonings sound logically. Therefore, the drawing is not precisely made.

These comments are not unnecessary, they allow us to find the error in figure 8, namely: the foot P of the altitude DP doesn't lay on the side BC (i.e. P is not between points B and C), but it lays on the extended side BC (fig. 9).

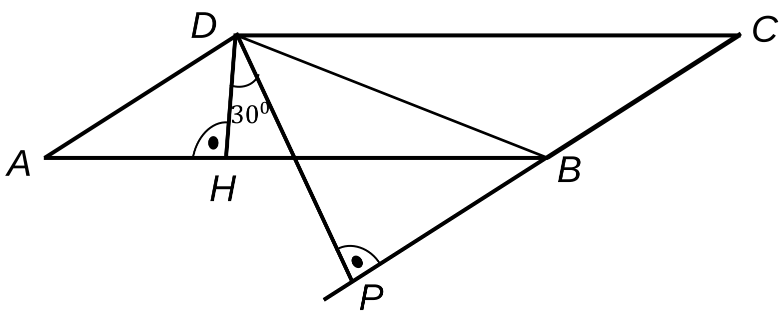

Fig. 9

Since $\measuredangle DCP = 30^0$, then $\measuredangle DAH = 30^0$ as well. Then, from ΔDHA follows that $DA = 2.DH = 4\ cm$ (a leg against angle of 30^0), and from ΔDPC, similarly follows that $DC = 2.DP = 8\ cm$. Then $AB = 8\ cm$, and $BC = 4\ cm$.

When one of the sides of the parallelogram is already found, the formula for the area of the parallelogram $S = AB.DH = BC.DP$ can be used to find the other side.

Different ways can also be used to find the diagonals BD and AC.

I way: Let consider ΔABD. Since AD, AB and $\measuredangle BAD$ are known, the cosine theorem can be used to find the diagonal DB. Similarly, for the diagonal AC the cosine theorem for ΔADC or for ΔABC can be used as well.

II way: Let consider the right-angle ΔBHD and apply the Pythagoras' theorem for it. To do this, we must find the length of HB first ($HB = AB - AH$), as AH is to be found from ΔAHD, for which $\measuredangle HAD = 30^0$ and $AD = 4$ см or $DH = 2$ см, or the Pythagoras' theorem can also be applied. Similar reasonings can be used to find the diagonal AC, and for this purpose an additional construction must be made – transfer the altitude DH to point C and continue the base AB until they intersect, for example in point M. From the congruence of ΔAHD and ΔBMC (Why?) follows that $BM = AH$, which means that AM is known. Then, using the Pythagoras' theorem for ΔAMC, the length of the diagonal AC is calculated.

As it can be seen, by applying different ideas for solving the problem, knowledge of the area of a parallelogram, cosine theorem, Pythagorean theorem, etc. is repeated.

We will complete the idea of repeating different knowledge by using problems that can be solved in different ways, as we discuss the following problem, which requires additional constructions.

Problem 7. In the right-angle triangle ΔABC with legs $BC = 6\ cm$ and $AB = 8\ cm$ the bisectors AA_1 and CC_1 are constructed. The points K and M are feet of the perpendiculars from the vertex B respectively to the lines AA_1 and CC_1. Find out the area of ΔBMK.

Solution:

I way: From the Pythagoras' theorem follows, that $AC = \sqrt{AB^2 + BC^2}$. Therefore, $AC = 10\ cm$. From $S_{ABC} = \frac{1}{2}BA.BC = \frac{1}{2}AC.h$ we find out, that $h = 4{,}8\ cm$. Let's continue the perpendiculars BK and BM until they intersect the hypotenuse AC in point E and point D, respectively. (fig. 10).

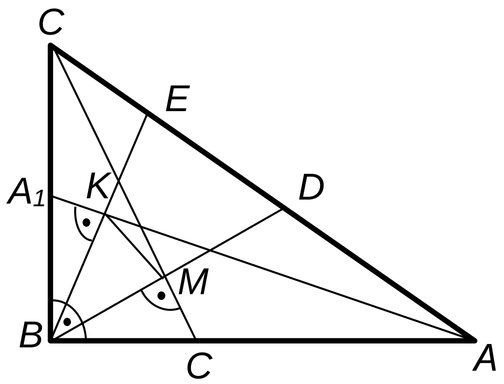

Fig. 10.

Let's consider ΔBCD, in which CM is a bisector of $\measuredangle C$. But it's given that $BM \perp CC_1$, i.e. $BM \perp CM$. Therefore CM is an altitude to BD in ΔBCD. Then $\Delta BCM \cong \Delta DCM$ (by SAS (II) rule). From the congruence of the triangles it follows, that $BC = CD$, i.e. that ΔBCD is isosceles. Therefore, CM is a median in ΔBCD, therefore point M is a midpoint of BD. (1)

In a simar way, for ΔBAE is proved, that $AB = AE$, and from the congruence of the triangles ΔBAK and ΔEAK follows, that point K is a midpoint of BE. (2)

From (1) and (2) follows, that MK is a middle segment of ΔBDE.

Since all middle segments in a given triangle divide it into 4 isosceles triangles, then $S_{BMK} = \frac{1}{4} S_{BDE}$. Therefore, in order to find the area of the triangle ΔBMK, it is enough to find the area of ΔBDE. It is fulfilled for it, that $S_{BDE} = \frac{1}{2}\, ED.h$ (the altitude h to the side ED in ΔBDE coincides with the altitude to the hypotenuse AC in ΔABC, for which it has already been found, that $h = 4{,}8\ cm$).

In order to find out the length of the side ED, we express it as follows

$ED = AC - CE - AD = AC - (CD - ED) - (AE - ED) =$

$= AC - (BC - ED) - (AB - ED) = 10 - 6 - 8 + 2.ED,$

from where $ED = 2.ED - 4$. Therefore $ED = 4\ cm$.

Then $S_{BED} = \frac{1}{2} ED.h = \frac{1}{2}.4.4{,}8 = 9{,}6\ cm^2$.

Therefore $S_{KBM} = \frac{1}{4} S_{EBD} = \frac{1}{4}.9{,}6 = 2{,}4\ cm^2$.

<u>II way</u>: It can be proved, that $KM \parallel ED$, then ΔBMK and ΔBDE are similar (by the lemma) and the ratio of their areas is equal to the square of the coefficient of

their similarity. As KM is a middle segment in $\Delta BDE,$ then $\frac{S_{BMK}}{S_{BDE}} = k^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ and

$S_{BMK} = \frac{1}{4} S_{BDE}$.

It can be proved, that ΔBAE and ΔBCD are isosceles, i.e. $AB = AE$ и $CB = CD$.

Then, from $AC = AE + CD - ED = AB + CB - ED$, after substituting the lengths of the given segments, it is found, that $ED = 4$ см. Since $sin\measuredangle A = \frac{BC}{AC} = \frac{h_{AC}}{AB}$, and

$AC = 10\ cm$, then $h_{AC} = \frac{AB.BC}{AC} = 4{,}8\ cm$.

Therefore $S_{BMK} = \frac{1}{4} S_{BDE} = \frac{1}{4}.\frac{1}{2}.ED.h_{AC} = 2{,}4\ cm^2$.

<u>III way</u>: Using the Pythagoras' theorem we find the length of the hypotenuse $AC = 10$ см. The segment ED is expressed again in the familiar way:

$ED = AC - CE - AD = AC - (CD - ED) - (AE - ED) =$

$= AC - (BC - ED) - (AB - ED)$,

from where it is received that $ED = 4\ cm$.

In the first way we have proved in details, that KM is a middle segment in ΔBDE. It follows, that $KM = \frac{1}{2} ED = \frac{1}{2}.4 = 2\ cm$.

As $\Delta BMK \sim \Delta BDE$ (by the lemma), then follows, that $\frac{h_{KM}}{h_{ED}} = \frac{KM}{ED}$;

$h_{KM} = \frac{1}{2} h_{ED} = \frac{1}{2} AB.sin\measuredangle A = \frac{1}{2} AB.\frac{BC}{AC} = 2{,}4\ cm$.

$S_{BMK} = \frac{1}{2} KM.h_{KM} = \frac{1}{2}.2.2{,}4 = 2{,}4\ cm^2$.

<u>IV way</u>: Since $S_{KBM} = \frac{1}{2} BK.BM.\sin\measuredangle KBM$, then we will find first BK, BM and $\sin\measuredangle KBM$. By Pythagoras' theorem we have that $AC = 10\,cm$. In ΔABC $\cos\measuredangle A = \frac{AB}{AC} = 0{,}8$, $\cos\measuredangle C = \frac{BC}{AC} = 0{,}6$.

For the isosceles ΔABE by cosine theorem is received, that:

$BE^2 = AB^2 + AE^2 - 2.AB.AE.\cos\measuredangle A$, whence $BE = 8\sqrt{0{,}4}$ and $BK = 4\sqrt{0{,}4}\ cm$.

In a similar way for the isosceles ΔBCD by cosine theorem is received, that:

$BD = 6\sqrt{0,8}\ cm$ и $BM = 3\sqrt{0,8}\ cm$.

To find ED we can use the following presentation $AC = AE + CE = AE + CD - ED = AB + CB - ED$. Therefore $ED = 4\ cm$.

Applying the cosine theorem for ΔBDE, we find $\cos\measuredangle DBE$. From $ED^2 = BE^2 + BD^2 - 2.BE.BD.\cos\measuredangle DBE$ we have that $cos\measuredangle DBE = \frac{BE^2+BD^2-ED^2}{2.BE.BD} = \frac{\sqrt{2}}{2}$. Then from $\cos\measuredangle DBE = \frac{\sqrt{2}}{2}$, follows, that $\measuredangle DBE = \measuredangle MBK = 45^0$. Therefore

$S_{KBM} = \frac{1}{2}BK.BM.sin\measuredangle MBK = \frac{1}{2}.4\sqrt{0,4}.3\sqrt{0,8}.sin\ 45^0 = 2,4\ cm^2$.

The content presented so far does not exhaust the possibilities for realization of repetition in teaching mathematics. As it is well known, the homework can also be used to achieve various aims of teaching mathematics, such as: confirming knowledge; preparation for learning a new concept; repetition of previously studied material. However, we will not present specific homework assignments here.

Through the illustrative examples of the repetition process mentioned in the article, the students' knowledge on the considered themes is expanded, their skills are developed, their mathematical knowledge is deepened, the connections between the separate knowledge are revealed, as a result of which the knowledge is acquired in a system. In such way, the systematic approach in teaching mathematics is accomplished.

In conclusion, we will note, that the student's ability to look for and find out different ways to solve a problem requires knowledge and repetition of different ideas for solving, but at the same time, it enriches his arsenal of such ideas. In addition, the discovery of different ways of solving problems and comparing them, for example in terms of their rationality, contributes to a deeper understanding of the theoretical knowledge used (definitions, theorems, properties, necessary conditions, sufficient conditions, necessary and sufficient conditions, etc.) and their systematization, which is the so-called systematic repetition and generalization. And they, in turn, have a positive impact on the lasting acquisition of knowledge acquired by the learner and the formation of flexible skills for its further application. Therefore, repetition must find a wide application in teaching mathematics at school.

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