2D AND 3D FINITE ELEMENT LOCALIZATION

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Introduction. The FEM is based on using of finite functions for approximation, which differ from zero only on small (finite) elements of the considered region [1,2]. To construct finite functions, the following operations are used:

- the computational domain $\Omega$ is divided by nodal points into $n$ elements;
- the selected function is displayed at these nodal points as discrete values;
- (basic) form functions $N_i$ are constructed, and their number exceeds by unit the number of elements.

The functions $N_i$ satisfy the localization and normalization conditions.

The analyzed function $f(\bar{x})$ is represented by a linear form $f(\bar{x}_i)$ from $N_i$

$$f(\bar{x}) = f(\bar{x}_0)N_0 + f(\bar{x}_1)N_1 + \ldots + f(\bar{x}_n)N_n = \sum_{i=0}^{n} f(\bar{x}_i) \cdot N_i$$  \hspace{1cm} (1)

If we expand $f(\bar{x})$ in a Fourier series with respect to $N_i$, then we again obtain formula (1). The accuracy of the approximation can be increased both by using more complex basis functions and by dividing the region into finer elements. Further, it is advisable to consider the question of the approximation accuracy of the modeled functions derivatives.

Problem's formulation. The purpose of the work is to distribute localization methods in FEM for finding the first and higher derivatives of functions, starting with simplexes in the computational domain.

Results. When passing from the solution of one-dimensional problems, we first consider the approximation of functions $\Phi(x, y)$ for differential operators in $R^2$. The finite element (simplex) is chosen in the form of a triangle with nodes at the vertices (Fig. 1).
On the diagram, a counterclockwise traversal of the nodes \( i \rightarrow j \rightarrow k \) is accepted, the coordinates of the nodes are indicated, starting from \((x_i, y_i)\), the values of the function in the nodes – from \( \Phi_i \).

Next, a linear form is defined by

\[
\Phi(x, y) = \Phi_i N_i + \Phi_j N_j + \Phi_k N_k,
\]

where

\[
N_i = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ x_j & y_j \\ x_k & y_k \end{vmatrix};
N_j = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ x_i & y_i \\ x_k & y_k \end{vmatrix};
N_k = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ x_i & y_i \\ x_j & y_j \end{vmatrix},
\]

\[
2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix},
\]

and \( A \) is the area of a triangle (simplex element).

Further, the form functions are written as:

\[
N_i(x, y) = a_i + b_i x + c_i y,
\]

where

\[
a_i = x_j y_k - x_k y_j; \quad b_i = y_j - y_k; \quad c_i = x_k - x_j.
\]

Then the first partial derivatives are found

\[
\frac{\partial \Phi}{\partial x} = \Phi_i \frac{\partial N_i}{\partial x} + \Phi_j \frac{\partial N_j}{\partial x} + \Phi_k \frac{\partial N_k}{\partial x},
\]

\[
\frac{\partial \Phi}{\partial y} = \Phi_i \frac{\partial N_i}{\partial y} + \Phi_j \frac{\partial N_j}{\partial y} + \Phi_k \frac{\partial N_k}{\partial y}.
\]

Taking into account (5), the approximating formulas are as follows

\[
\frac{\partial \Phi}{\partial x} = \Phi_i b_i \Phi_j b_j \Phi_k b_k,
\]
\[
\frac{\partial \Phi}{\partial y} = \Phi_i c_i + \Phi_j c_j + \Phi_k c_k. \tag{8}
\]

Further \( \frac{\partial \Phi}{\partial x} \) and \( \frac{\partial \Phi}{\partial y} \) are written as linear form by \( N_i, N_j, N_k \).

The values of the form functions calculated by formulas (2) and (3) satisfy the normalization conditions \( \forall \bar{x}_j \in \Omega \sum_{i=0}^{n} N_i(\bar{x}_j) = 1 \). The linear form is also Fourier series.

\[
\frac{\partial \Phi}{\partial x} = \left( \Phi_i b_i + \Phi_j b_j + \Phi_k b_k \right) N_i + \left( \Phi_i b_i + \Phi_j b_j + \Phi_k b_k \right) N_j + \left( \Phi_i b_i + \Phi_j b_j + \Phi_k b_k \right) N_k \tag{9}
\]

Then, using formula (9), we determine

\[
\frac{\partial^2 \Phi}{\partial x^2} = \left( \Phi_i b_i + \Phi_j b_j + \Phi_k b_k \right) b_i + \left( \Phi_i b_i + \Phi_j b_j + \Phi_k b_k \right) b_j + \left( \Phi_i b_i + \Phi_j b_j + \Phi_k b_k \right) b_k. \tag{10}
\]

Analogically

\[
\frac{\partial^2 \Phi}{\partial y^2} = \left( \Phi_i c_i + \Phi_j c_j + \Phi_k c_k \right) c_i + \left( \Phi_i c_i + \Phi_j c_j + \Phi_k c_k \right) c_j + \left( \Phi_i c_i + \Phi_j c_j + \Phi_k c_k \right) c_k. \tag{11}
\]

You can use the matrix notation of formulas (10) and (11)

\[
\frac{\partial^2 \Phi}{\partial x^2} = \begin{pmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_i b_j & b_j^2 & b_j b_k \\ b_i b_k & b_j b_k & b_k^2 \end{pmatrix} \begin{pmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{pmatrix}, \tag{12}
\]

\[
\frac{\partial^2 \Phi}{\partial y^2} = \begin{pmatrix} c_i^2 & c_i c_j & c_i c_k \\ c_i c_j & c_j^2 & c_j c_k \\ c_i c_k & c_j c_k & c_k^2 \end{pmatrix} \begin{pmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{pmatrix}. \tag{13}
\]

Laplace operator after addition of derivatives by formulas (12) and (13) is
\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \begin{bmatrix}
(b_i^2 + c_i^2) & bi b_j + c_i c_j & bi bk + c_i c_k \\
bi bj + c_i c_j & b_j^2 + c_j^2 & b j bk + c_j c_k \\
bi bk + c_i c_k & bj bk + c_j c_k & b_k^2 + c_k^2
\end{bmatrix} \cdot \begin{bmatrix}
\Phi_i \\
\Phi_j \\
\Phi_k
\end{bmatrix}.
\] (14)

The first factor in (14) is naturally represented as a matrix product

\[
\begin{bmatrix}
(b_i^2 + c_i^2) & bi b_j + c_i c_j & bi bk + c_i c_k \\
bi bj + c_i c_j & b_j^2 + c_j^2 & b j bk + c_j c_k \\
bi bk + c_i c_k & bj bk + c_j c_k & b_k^2 + c_k^2
\end{bmatrix} = \begin{bmatrix}
bi \\
bi b_j \\
bi bk
\end{bmatrix} \cdot \begin{bmatrix}
c_i \\
c_j \\
c_k
\end{bmatrix}.
\] (15)

When passing to the three-dimensional space \( \mathbb{R}^3 \), it is natural to use a tetrahedron as a simplex (Fig. 2). Let the coordinates of the vertices-nodes of the elements are \((x_i, y_i, z_i), \ldots, (x_L, y_L, z_L)\), and the values of the function at the nodes are \(\Phi_i, \ldots, \Phi_L\)

\[
\text{Fig. 2. Finite element diagram in } \mathbb{R}^3
\]

Next, the linear form is used

\[
f(x, y, z) = \Phi_i N_i + \Phi_j N_j + \Phi_k N_k + \Phi_L N_L.
\] (16)

The volume of a simplex-tetrahedron equals to value \(\frac{C}{6}\), where the value of \(C\) is determined by the formula
\[
C = \begin{vmatrix}
1 & x_i & y_i & z_i \\
1 & x_j & y_j & z_j \\
1 & x_k & y_k & z_k \\
1 & x_L & y_L & z_L \\
\end{vmatrix}.
\] (17)

Form functions are found using determinants
\[
N_i = \frac{1}{C} \begin{vmatrix}
1 & x & y & z \\
1 & x_j & y_j & z_j \\
1 & x_k & y_k & z_k \\
1 & x_L & y_L & z_L \\
\end{vmatrix} ; N_j = \frac{1}{C} \begin{vmatrix}
1 & x_i & y_i & z_i \\
1 & x_j & y_j & z_j \\
1 & x_k & y_k & z_k \\
1 & x_L & y_L & z_L \\
\end{vmatrix}.
\] (18)

Values of \( N_k, N_L \) are calculated similarly
\[
\frac{\partial f}{\partial x} = \Phi_i \frac{\partial N_i}{\partial x} + \Phi_j \frac{\partial N_j}{\partial x} + \Phi_k \frac{\partial N_k}{\partial x} + \Phi_L \frac{\partial N_L}{\partial x} ;
\]
\[
\frac{\partial f}{\partial y} = \Phi_i \frac{\partial N_i}{\partial y} + \Phi_j \frac{\partial N_j}{\partial y} + \Phi_k \frac{\partial N_k}{\partial y} + \Phi_L \frac{\partial N_L}{\partial y} ;
\]
\[
\frac{\partial f}{\partial z} = \Phi_i \frac{\partial N_i}{\partial z} + \Phi_j \frac{\partial N_j}{\partial z} + \Phi_k \frac{\partial N_k}{\partial z} + \Phi_L \frac{\partial N_L}{\partial z} .
\] (19)

Then the second derivative is approximated
\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial N_k}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial N_L}{\partial x} ,
\] (22)

where \( \frac{\partial f}{\partial x} \) is determined by formula (18).

Similarly \( \frac{\partial^2 f}{\partial y^2} \) and \( \frac{\partial^2 f}{\partial z^2} \) are found. The problems of constructing systems of algebraic equations for a group of finite elements from triangles or tetrahedrons do not present any special difficulties and are considered in [1].

For various boundary value problems, FEM and the theory of difference schemes provide an equivalent approximation of differential operators. It is natural to require that the problems obtained after the approximation be solvable. In addition, their solution with an increase in the number of form functions should approach the solution of the original variation problem or the problem for a differential equation.

Attention should be paid to a very significant fact. At displaying of continuous function on the computational domain in the form of discrete values \( f(\tilde{x}_i) \) with a carrier representing a one-point set \( u(\tilde{x}_i, \epsilon), \epsilon \to 0 \), the transition to the “Theory of distributions (generalized functions)” section of functional analysis occurs. According
to [6], the equation
\[
f(x) = f(x_0)N_0 + f(x_1)N_1 + \ldots + f(x_n)N_n = \sum_{i=0}^{n} f(x_i) \cdot N_i
\]
gives the partition function of unity in the class \( C^k_0 \), where \( k \) denotes the continuity of derivatives up to the \( k \)-th order, and \( k \) is equal to the number of nodes on a finite element minus one and coincides with (1).

**Conclusions.** Formula (1) defines a Fourier series in functions that satisfy the conditions of continuity and localization, which gives the best approximation of the functions. The use of the compactness of the basis functions, which vanish everywhere, except for a fixed number of elementary subdomains into which the given domain is divided, contributes to the stability of the numerical solution.

The implementation of the finite element method can be considered as a method of sequential numerical differentiation of Fourier series with respect to continuous functions that satisfy the principle of localization, that is, cutoff functions [6]. Trial functions (shape functions in the finite element method) provide convergence if the linear form of the grid functions is a distribution, that is, in aggregate, it represents partitions of unity \( \text{supp} f(x_i) \subset \text{supp} N_i = u(x, \varepsilon) \).

Next, one should proceed to obtaining estimates for the deviations of the approximate solution in the space of finite elements from the exact solution of the problem.

**References:**